NORTH-HOLLAND

A Bound on the Exponent of a Primitive Matrix Using Boolean Rank*

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Dedicated to Professor John Maybee on the occasion of his 65'th birthday.

Submitted by Richard A. Brualdi

ABSTRACT

We present a bound on the exponent $\exp(A)$ of an $n \times n$ primitive matrix A in terms of its boolean rank $b = b(A)$; namely $\exp(A) \leqslant (b-1)^2 + 2$. Further, we show that for each $2 \leqslant b \leqslant n-1$, there is an $n \times n$ primitive matrix A with $b(A) = b$ such that $\exp(A) = (b-1)^2 + 2$, and we explicitly describe all such matrices. The new bound is compared with a well-known bound of Dulmage and Mendelsohn, and with a conjectured bound of Hartwig and Neumann. Several open problems are posed.

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1. INTRODUCTION

A real $n \times n$ (entrywise) nonnegative matrix A is *primitive* if one of its powers, A^k, has all positive entries for some integer $k \geqslant 1$. The smallest such k is called the *exponent* of A, and is denoted by $\exp(A)$. Since a primitive matrix A can have no zero *line* (i.e. row or column), we see that each entry of A^m is positive whenever $m \geqslant \exp(A)$. There is an extensive literature on exponents of primitive matrices, and a good survey of results and references can be found in Brualdi and Ryser [1, Section 3.5].

It is clear that neither the primitivity nor the exponent of a primitive matrix depends on the size of the nonzero entries in a nonnegative matrix A; they only depend on the location of the nonzero entries within A. Consequently, it is natural to make the following definition. Two $n \times n$ nonnegative matrices are said to be *combinatorially equivalent* if their nonzero entries are in precisely the same locations. Thus any matrix B which is combinatorially equivalent to a primitive matrix A is also primitive, and $\exp(B) = \exp(A)$.

A number of authors have worked on obtaining upper bounds on the exponent of a primitive matrix. The earliest such bound is due to Wielandt [8], and we summarize his result below (a proof can be found in [1]).

PROPOSITION 1.1. *If A is an $n \times n$ primitive matrix, then*

$$\exp(A) \leqslant (n-1)^2 + 1. \tag{1.1}$$

Further, equality holds in (1.1) *if and only if there is a permutation matrix P such that PAP^t is combinatorially equivalent to W_n, where*

$$W_1 = [1], \qquad W_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$W_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \qquad \textit{for} \quad n \geqslant 3.$$

Wielandt's bound was later generalized by Dulmage and Mendelsohn [2]. They showed that if A is an $n \times n$ primitive matrix, and s is the smallest integer $k \geqslant 1$ such that A^k has a nonzero diagonal entry, then

$$\exp(A) \leqslant n + s(n-2). \tag{1.2}$$

We will provide another upper bound on the exponent in terms of the boolean rank of a nonnegative matrix, a concept which we now discuss. Given a nonzero $m \times n$ nonnegative matrix A, we define its *boolean rank* $b(A)$, to be the smallest integer k such that for some nonnegative matrices F and G with F $m \times k$ and G $k \times n$, A is combinatorially equivalent to the product FG. The boolean rank of the zero matrix is defined to be 0. (In Section 2, we discuss the notion of boolean rank in more detail.)

It follows from Theorem 3.1 below that for any $n \times n$ primitive matrix A, if $b(A) = b$ and $1 < b < n$, then

$$\exp(A) \leqslant (b-1)^2 + 2. \tag{1.3}$$

In Theorem 3.2, we show that the bound (1.3) can be attained for each such b, and we give an explicit characterization of those matrices that do so. That extends Proposition 1.1 above, which characterizes the matrices realizing equality in (1.1) (note that those matrices all have boolean rank n).

In Section 4, we compare Dulmage and Mendelsohn's bound (1.2) with (1.3), showing by example that sometimes it is better and sometimes it is worse. Examples illustrating other points of interest are also presented there. Finally, we suggest some open problems in Section 5.

Throughout, our results are obtained by using the machinery of (0, 1) boolean matrices. In Section 2 below, we give the relevant definitions and basic properties of these matrices.

2. PRELIMINARIES

Given a nonnegative matrix A, we define its *pattern* A_+, to be the (0, 1) matrix which is combinatorially equivalent to A. In computing sums and products of (0, 1) matrices and vectors, we will use *boolean arithmetic*—that is, the usual arithmetic except that $1 + 1 = 1$. It now follows that for any $m \times n$ nonnegative matrices A and B, and any $n \times k$ nonnegative matrix C, we have $(A+B)_+ = A_+ + B_+$ and $(AC)_+ = A_+ C_+$. The (0, 1) matrices under this boolean arithmetic are called (0, 1) *boolean* (or just *boolean*) matrices. The book of Kim [6] gives an extensive treatment of boolean matrix theory.

The definitions of primitivity and the exponent of a primitive matrix extend quite naturally to (0, 1) boolean matrices. Specifically, if B is an $n \times n$ (0, 1) boolean matrix, then B is *primitive* if one of its powers, B^k, has no zero entry, in which case, its *exponent*, $\exp(B)$, is the smallest such integer k. Thus we see that a nonnegative matrix A is primitive if and only if its pattern A_+ (a boolean matrix) is, and that when both are primitive, $\exp(A) = \exp(A_+)$. *For this reason, we will henceforth confine our investigation of the exponent to* (0, 1) *boolean matrices.*

Notice that for any nonnegative matrix A, the boolean ranks of A and A_+ are the same. Since we will be concerning ourselves with (0, 1) boolean matrices (such as A_+) throughout the sequel, we will take a brief look at the boolean rank of a (0, 1) boolean matrix. If B is an $m \times n$ (0, 1) boolean matrix, it is not difficult to see that $b(B)$ is the smallest integer k such that there are (0, 1) boolean matrices X and Y, with X $m \times k$ and Y $k \times n$, such that $B = XY$. (Indeed, this is the standard definition of boolean rank for a (0, 1) boolean matrix; see Kim [6] for example, who calls it "Schein rank.") If $b(B) = b$, X is $m \times b$, Y is $b \times n$, and the product XY equals B, we will say that $B = XY$ is a *(boolean) rank factorization of* B. The boolean rank of B can also be thought of as the minimum number of matrices of boolean rank 1 [i.e. outer products of the form xy^t where x is a (0, 1) boolean m-vector and y is a (0, 1) boolean n-vector] that sum to B.

A set of positions $\{(i_k, j_k) : 1 \leqslant k \leqslant l\}$ is said to be *isolated* (in the matrix B) if the following conditions hold:

(1) for each $1 \leqslant k \leqslant l$, the (i_k, j_k) entry of B is 1,
(2) no two positions in the set are in the same row or column of B, and
(3) no two positions in the set are in the same 2×2 all ones submatrix of B.

Such a set of positions is referred to as a *set of isolated ones (in* B*)* (see Gregory and Pullman [3]). Using our second interpretation of boolean rank, it follows that $b(B)$ is always bounded below by the cardinality of any set of isolated ones in B. We will occasionally need this fact in order to estimate the boolean rank from below; in particular, the following result will be useful.

PROPOSITION 2.1. *Suppose that* B *is an* $m \times n$ (0, 1) *boolean matrix, and that* $b(B) \leqslant k$. *If* B *has a* k*-element set of isolated ones, then* $b(B) = k$.

Throughout the sequel, we will use the following notation and terminology. Given an $m \times n$ matrix A, we will denote its entry in the (i, j) spot by A_{ij}, its ith row by $A_{i\cdot}$, and its jth column by $A_{\cdot j}$. If B is another $m \times n$ matrix, we say that B *is dominated by* A, and write $B \leqslant A$, if $B_{ij} \leqslant A_{ij}$ for all i and j; the same terminology and notation will be used for vectors. We denote the $m \times n$ all ones matrix by $J_{m,n}$ (and by J_n if $m = n$), the $m \times n$ all zeros matrix by $0_{m,n}$, the all ones n-vector by j_n, the $n \times n$ identity matrix by I_n, and its ith column by $e_i(n)$. The subscripts m and n will be omitted whenever their values are clear from the context.

3. A BOUND ON THE EXPONENT

Our bound on the exponent rests on the following observations. The first was noted by Shao [7], and the second will be discussed further in Section 4.

PROPOSITION 3.1. *Suppose that X and Y are $n \times m$ and $m \times n$ (0, 1) boolean matrices (respectively) and that neither has a zero line. Then*

(a) *XY is primitive if and only if YX is primitive, and*
(b) *if XY and YX are primitive, then*

$$|\exp(XY) - \exp(YX)| \leqslant 1.$$

Proof. Suppose that XY is primitive, with $\exp(XY) = k$. Then $(YX)^{k+1} = Y(XY)^k X = YJ_n X = J_m$, the last equality following from the fact that neither X nor Y has a zero line. Thus YX is primitive and $\exp(YX) \leqslant \exp(XY) + 1$. The result follows by exchanging the roles of X and Y. ∎

It follows from Proposition 3.1 that if $X_1, X_2, \dots, X_d$ are $n_1 \times n_2, n_2 \times n_3, \dots, n_d \times n_1$ (0, 1) boolean matrices (respectively) with no zero lines, then

(1) each of the cyclic products $X_1 X_2 \cdots X_d$, $X_2 X_3 \cdots X_d X_1, \dots,$ $X_d X_1 X_2 \cdots X_{d-1}$ is primitive if any one of them is, and
(2) in the case that each is primitive, the exponents of any two differ by at most 1.

These statements also follow from a result of Dulmage and Mendelsohn (see [1, p. 78, Exercise 5]).

Now we apply the proposition to get an upper bound on the exponent of a primitive matrix.

THEOREM 3.1. *Suppose that $n \geqslant 2$ and that B is an $n \times n$ primitive (0, 1) boolean matrix with $b(B) = b$. Then*

$$\exp(B) \leqslant (b-1)^2 + 2. \tag{3.1}$$

Proof. Let $B = XY$ be a boolean rank factorization of B. Then X is $n \times b$, Y is $b \times n$, and neither has a zero line. By Proposition 3.1, the $b \times b$ matrix YX is primitive, and $\exp(B) \leqslant \exp(YX) + 1$. Wielandt's inequality (1.1) implies that $\exp(YX) \leqslant (b-1)^2 + 1$, and so (3.1) follows. ∎

From (1.1) we see that no matrix of boolean rank n can realize equality in (3.1). Further, since the only $n \times n$ primitive $(0, 1)$ boolean matrix of boolean rank 1 is J_n, we see that no matrix of boolean rank 1 can realize equality in (3.1). However, we will show that for each $1 < b < n$, there is an $n \times n$ matrix of boolean rank b realizing equality in (3.1). In order to do so, we need to make some observations about the matrix W_n of Proposition 1.1.

PROPOSITION 3.2. *If $n \geqslant 2$, the only zero entry in $W_n^{(n-1)^2}$ occurs in the (n, n) position.*

Proof. As always, we are thinking of W_n as a $(0, 1)$ boolean matrix. The result is immediate if $n = 2$. For $n \geqslant 3$, a straightforward proof by induction on k shows that for $1 \leqslant k \leqslant n-1$,

$$W_n^k = \left[\begin{array}{c|c} U & I_{n-k} \\ \hline V & 0_{k,n-k} \end{array}\right],$$

where the $(n-k) \times k$ matrix U is all 0 except for a 1 in the $(n-k, 1)$ position, and the $k \times k$ matrix V has 1's on the diagonal and superdiagonal, and 0's elsewhere. It follows that $W_n^n = W_n + I$, and hence that $W_n^{(n-1)^2} = W_n(W_n + I)^{n-2} = \sum_{k=1}^{n-1} W_n^k$. The formulae for W_n^k, $1 \leqslant k \leqslant n-1$, now yield the result. ∎

Next, we give a characterization of the matrices for which equality holds in (3.1).

PROPOSITION 3.3. *Suppose that B is an $n \times n$ $(0, 1)$ boolean matrix with $2 \leqslant b = b(B) \leqslant n-1$. Then B is primitive with* $\exp(B) = (b-1)^2 + 2$ *if and only if B has a boolean rank factorization $B = XY$, where X and Y have the following properties*:

(i) $YX = W_b$, *and*
(ii) *some row of X is $e_b^t(b)$ and some column of Y is $e_b(b)$.*

Proof. First, suppose that B is primitive with $\exp(B) = (b-1)^2 + 2$, and that $B = \overline{X}\overline{Y}$ is a boolean rank factorization of B. By Proposition 3.1, $\overline{Y}\overline{X}$ is primitive and $\exp(\overline{Y}\overline{X}) \geqslant (b-1)^2 + 1$. Thus by Proposition 1.1,

$\exp(\overline{Y}\overline{X}) = (b-1)^2 + 1$, and so there is a permutation matrix P such that $P\overline{Y}\overline{X}P^t = W_b$. Letting $X = \overline{X}P^t$ and $Y = P\overline{Y}$, we find that $B = XY$ is a rank factorization of B, and that $YX = W_b$. Thus X and Y satisfy (i).

Since B is primitive, it follows that $\sum_{i=1}^{b} X_{\cdot i} = j_n = \sum_{i=1}^{b} Y_{i\cdot}^t$. Further, since $\exp(B) = (b-1)^2 + 2$, the matrix $B^{(b-1)^2+1}$ must have a zero entry, say in the (p,q) position. But by Proposition 3.2, $B^{(b-1)^2+1} = XW_b^{(b-1)^2}Y = XZY$, where Z is the $b \times b$ matrix whose only zero entry is in the (b,b) position. Thus

$$B^{(b-1)^2+1} = \left[J_{n,b-1} \;\middle|\; \sum_{i=1}^{b-1} X_{\cdot i} \right] Y = j_n \left(\sum_{i=1}^{b-1} Y_{i\cdot} \right) + \left(\sum_{i=1}^{b-1} X_{\cdot i} \right) Y_{b\cdot},$$

and hence $\sum_{i=1}^{b-1} Y_{iq} + \sum_{i=1}^{b-1} Y_{pi} Y_{bq} = 0$. So the first $b-1$ entries in $Y_{\cdot q}$ are zero, and since Y has no zero lines, we see that $Y_{\cdot q}$ must equal $e_b(b)$. Hence the first $b-1$ entries of $X_{p\cdot}$ are zero also, and we find that $X_{p\cdot}$ must equal $e_b^t(b)$. Consequently, X and Y satisfy (ii).

Finally, suppose that $B = XY$ is a rank factorization of B and that X and Y satisfy (i) and (ii). Then B is primitive by Proposition 3.1(a), and $\exp(B) \leqslant (b-1)^2 + 2$ by Proposition 3.1(b) and Theorem 1.1. But it follows from Proposition 3.2 and conditions (i) and (ii) that $B^{(b-1)^2+1}$ has a zero entry, and so we conclude that $\exp(B) = (b-1)^2 + 2$. ∎

Our last result of this section will reinterpret conditions (i) and (ii) of Proposition 3.3 to show that if B yields equality in (3.1), then B is one of 14 basic types of matrices.

THEOREM 3.2. *Suppose B is an $n \times n$ $(0,1)$ boolean matrix with $b(B) = b$, and that $2 \leqslant b \leqslant n-1$. Then B is primitive with* $\exp(B) = (b-1)^2 + 2$ *if and only if there is a permutation matrix Q such that QBQ^t has one of the forms displayed in Table* 1 *(if $3 \leqslant b \leqslant n-1$) or Table* 2 *(if $b = 2$).*

In Table 1 the rows and columns of $M_1, \dots, M_7$ are partitioned conformally, so that each diagonal block is square, and the top left hand submatrix common to each has b blocks in its partitioning. Likewise, in Table 2 the rows and columns of $N_1, \dots, N_7$ are partitioned conformally.

Proof. Suppose that B is primitive, $b \geqslant 3$, and $\exp(B) = (b-1)^2 + 2$; by Proposition 3.3, there is a boolean rank factorization of B as $B = XY$ such that

(i) $YX = W_b$, and
(ii) some row of X is $e_b^t(b)$ and some column of Y is $e_b(b)$.

TABLE 1
$b \geqslant 3$

$$M_1 = \left[\begin{array}{cccccc|c} 0 & J & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J & 0 & J \\ 0 & 0 & \cdots & 0 & 0 & J & 0 \\ J & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline J & 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right] \qquad M_2 = \left[\begin{array}{cccccc|c} 0 & J & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J & 0 & J \\ 0 & 0 & \cdots & 0 & 0 & J & J \\ J & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline J & 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right]$$

$$M_3 = \left[\begin{array}{cccccc|c} 0 & J & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J & 0 & J \\ 0 & 0 & \cdots & 0 & 0 & J & 0 \\ J & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline J & 0 & \cdots & 0 & 0 & J & 0 \end{array}\right] \qquad M_4 = \left[\begin{array}{cccccc|cc} 0 & J & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J & 0 & J & J \\ 0 & 0 & \cdots & 0 & 0 & J & 0 & J \\ J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \hline J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$M_5 = \left[\begin{array}{cccccc|cc} 0 & J & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J & 0 & J & J \\ 0 & 0 & \cdots & 0 & 0 & J & 0 & 0 \\ J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \hline J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ J & 0 & \cdots & 0 & 0 & J & 0 & 0 \end{array}\right] \qquad M_6 = \left[\begin{array}{cccccc|cc} 0 & J & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J & 0 & J & J \\ 0 & 0 & \cdots & 0 & 0 & J & J & 0 \\ J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \hline J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ J & 0 & \cdots & 0 & 0 & J & J & 0 \end{array}\right]$$

$$M_7 = \left[\begin{array}{cccccc|ccc} 0 & J & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J & 0 & J & J & J \\ 0 & 0 & \cdots & 0 & 0 & J & 0 & J & 0 \\ J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ J & 0 & \cdots & 0 & 0 & J & 0 & J & 0 \end{array}\right]$$

Since B is primitive, X has no zero row, and so each column of Y is dominated by a column of W_b. Similarly, each row of X is dominated by a row of W_b. Thus, each column of Y is in the set $\mathscr{C} = \{e_1(b), e_2(b), \ldots, e_b(b), u\}$, where $u = e_{b-1}(b) + e_b(b)$, and each row of X is in the set $\mathscr{R} = \{e_1^t(b), \ldots, e_b^t(b), v^t\}$, where $v = e_1(b) + e_b(b)$.

Next, we note that for each $1 \leqslant i \leqslant b$, the outer product $Y_{\cdot i} X_{i \cdot}$ is dominated by W_b. Since each such $Y_{\cdot i}$ and $X_{i \cdot}$ must be in $\mathscr{C}$ and $\mathscr{R}$ respectively, we find that $(Y_{\cdot i}, X_{i \cdot})$ must be one of the following pairs:

TABLE 2
$b = 2$

$$N_1 = \left[\begin{array}{cc|c} 0 & J & 0 \\ J & 0 & J \\ \hline J & 0 & J \end{array}\right] \qquad N_2 = \left[\begin{array}{cc|c} 0 & J & J \\ J & 0 & J \\ \hline J & 0 & J \end{array}\right] \qquad N_3 = \left[\begin{array}{cc|c} 0 & J & 0 \\ J & 0 & J \\ \hline J & J & J \end{array}\right]$$

$$N_4 = \left[\begin{array}{cc|cc} 0 & J & 0 & J \\ J & 0 & J & J \\ \hline J & 0 & J & J \\ J & 0 & J & J \end{array}\right] \qquad N_5 = \left[\begin{array}{cc|cc} 0 & J & 0 & 0 \\ J & 0 & J & J \\ \hline J & 0 & J & J \\ J & J & J & J \end{array}\right] \qquad N_6 = \left[\begin{array}{cc|cc} 0 & J & J & 0 \\ J & 0 & J & J \\ \hline J & 0 & J & J \\ J & J & J & J \end{array}\right]$$

$$N_7 = \left[\begin{array}{cc|ccc} 0 & J & 0 & J & 0 \\ J & 0 & J & J & J \\ \hline J & 0 & J & J & J \\ J & 0 & J & J & J \\ J & J & J & J & J \end{array}\right]$$

(e_i, e_{i+1}^t), $1 \leqslant i \leqslant b-1$, (e_{b-1}, e_1^t), (e_b, e_1^t), (u, e_1^t), and (e_{b-1}, v^t). For each $1 \leqslant i \leqslant b-2$, e_i is a column of W_b that contains the only nonzero entry of row i. Thus, for each such i, $(e_i, e_{i+1}^t) = (Y_{\cdot k_i}, X_{k_i \cdot})$ for some k_i. Further, from (ii), some $X_{j\cdot}$ is e_b^t, so $(e_{b-1}, e_b^t) = (Y_{\cdot k_{b-1}}, X_{k_{b-1}\cdot})$ for some k_{b-1}. Similarly, (ii) implies that some $Y_{\cdot j}$ is e_b, so that $(e_b, e_1^t) = (Y_{\cdot k_b}, X_{k_b \cdot})$ for some k_b. Finally, some outer product $Y_{\cdot j} X_{j\cdot}$ must have a 1 in the $(b-1, 1)$ spot, and hence for some k_{b+1}, $(Y_{\cdot k_{b+1}}, X_{k_{b+1}\cdot})$ is one of (e_{b-1}, e_1^t), (u, e_1^t), or (e_{b-1}, v^t).

From the above considerations, it follows that there is an $n \times n$ permutation matrix Q such that

$$YQ^t = \left[\bar{Y} \mid \tilde{Y}\right] \quad \text{and} \quad QX = \left[\begin{array}{c} \bar{X} \\ \hline \tilde{X} \end{array}\right],$$

where

$$\bar{Y} = \left[e_1 j_{n_1}^t \mid e_2 j_{n_2}^t \mid \cdots \mid e_b j_{n_b}^t\right] \quad \text{and} \quad \bar{X} = \left[\begin{array}{c} j_{n_1} e_2^t \\ \hline j_{n_2} e_3^t \\ \hline \vdots \\ \hline j_{n_{b-1}} e_b^t \\ \hline j_{n_b} e_1^t \end{array}\right]$$

for some $n_1, \ldots, n_b \geqslant 1$, and where each $(\tilde{Y}_{\cdot i}, \tilde{X}_{i\cdot})$ is one of (e_{b-1}, e_1^t), (u, e_1^t), or (e_{b-1}, v^t). Thus the pair $\tilde{Y}$ and $\tilde{X}$ can be taken to be one of the following pairs of matrices:

$$\tilde{Y}_1 = e_{b-1} j_{m_1}^t, \ \tilde{X}_1 = j_{m_1} e_1^t \text{ for some } m_1 \geqslant 1;$$

$$\tilde{Y}_2 = u j_{m_2}^t, \ \tilde{Y}_2 = j_{m_2} e_1^t \text{ for some } m_2 \geqslant 1;$$

$$\tilde{Y}_3 = e_{b-1} j_{m_3}^t, \ \tilde{X}_3 = j_{m_3} v^t \text{ for some } m_3 \geqslant 1;$$

$$\tilde{Y}_4 = \left[e_{b-1} j_{m_4}^t \mid u j_{p_4}^t \right], \ \tilde{X}_4 = \left[\frac{j_{m_4} e_1^t}{j_{p_4} e_1^t} \right] \text{ for some } m_4, p_4 \geqslant 1;$$

$$\tilde{Y}_5 = \left[e_{b-1} j_{m_5}^t \mid e_{b-1} j_{p_5}^t \right], \ \tilde{X}_5 = \left[\frac{j_{m_5} e_1^t}{j_{p_5} v^t} \right] \text{ for some } m_5, p_5 \geqslant 1;$$

$$\tilde{Y}_6 = \left[u j_{m_6}^t \mid e_{b-1} j_{p_6}^t \right], \ \tilde{X}_6 = \left[\frac{j_{m_6} e_1^t}{j_{p_6} v^t} \right] \text{ for some } m_6, p_6 \geqslant 1;$$

$$\tilde{Y}_7 = \left[e_{b-1} j_{m_7}^t \mid u j_{p_7}^t \mid e_{b-1} j_{q_7}^t \right], \ \tilde{X}_6 = \begin{bmatrix} j_{m_7} e_1^t \\ \hline j_{p_7} e_1^t \\ \hline j_{q_7} v^t \end{bmatrix} \text{ for some } m_7, p_7, q_7 \geqslant 1.$$

It is now readily verified that

$$\left[\frac{\tilde{X}}{\tilde{X}_i} \right] \left[\tilde{Y} \mid \tilde{Y}_i \right] = M_i \qquad \text{for} \quad 1 \leqslant i \leqslant 7,$$

so that QBQ^t is one of the matrices in Table 1. Minor modifications of this argument show that if $b = 2$, B is primitive, and $\exp(B) = (2-1)^2 + 2 = 3$, then B is permutationally similar to one of the matrices in Table 2.

Finally, since the rank factorization

$$M_i = \left[\frac{\tilde{X}}{\tilde{X}_i} \right] \left[\tilde{Y} | \tilde{Y}_i \right]$$

satisfies conditions (i) and (ii) of Proposition 3.3, we see that each M_i is primitive and $\exp(M_i) = (b - 1)^2 + 2$; a similar argument applies to the N_i's in Table 3.2. ■

4. COMPARISONS

If B is an $n \times n$ (0, 1) boolean matrix and $b(B) = n$, then obviously Wielandt's bound (1.1) on $\exp(B)$ is sharper than that of (3.1). However, if $b(B) \leqslant n - 1$ we see that (3.1) is an improvement on (1.1). It is natural to wonder how (3.1) compares with Dulmage and Mendelsohn's bound (1.2); the example below shows that they are not comparable in general.

EXAMPLE 4.1. For any $n \geqslant 3$, let A be the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 1 & \cdots & 1 & 1 \end{bmatrix},$$

and note that $\exp(A) = 2$. Since A has exactly two distinct rows, we find that $b(A) = 2$. Thus the right hand side of (3.1) is equal to 3. Since A has a one on the diagonal (indeed several), the right hand side of (1.2) is $2n - 2$. Thus for the matrix A, (3.1) is a sharper bound than (1.2).

Next, for any $n \geqslant 5$, let B be the $n \times n$ matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

A straightforward calculation shows that $\exp(B) = 3$. Since the n − 1st and n − 2nd rows of B are the same, $b(B) \leqslant n - 1$. Now B has a set of isolated ones in positions $(1,1), (2,2), \ldots, (n-2, n-2), (n, n-1)$, and so $b(B) = n - 1$ by Proposition 2.1. Thus the right side of (3.1) is equal to $(n - 2)^2 + 2$.

However, the right side of (1.2) is equal to $2n - 2$, since B has a one on the diagonal, so we see that for B, (1.2) is sharper than (3.1).

Finally, we remark that if $2 \leqslant n \leqslant 4$ and $b \leqslant n - 1$, then $n + s(n - 2) \geqslant 2n - 2 \geqslant (n-2)^2 + 2 \geqslant (b-1)^2 + 2$ for any $s \geqslant 1$, so that for such n and b, (1.2) is never tighter than (3.1).

It is interesting to examine the bound (3.1) in connection with some recent work on exponents. In a working paper, Hartwig [4] conjectured that if M is a primitive matrix, then

$$\exp(M) \leqslant d^2 + 1, \tag{4.1}$$

where $d = d(M)$ is the smallest integer k such that $I + M + M^2 + \cdots + M^k = J$. Now d is the diameter of the directed graph associated with M, and since the positions in M corresponding to the arcs of a shortest directed path must be isolated in M, it follows that $d(M) \leqslant b(M)$ for any primitive M. By looking at the directed graph associated with the matrix A of Example 4.1, we find that $d(A) = 2$. But $b(A) = 2$ as well, so the bound (3.1) is better than the conjectured bound (4.1) for this example. However, if M is a primitive matrix such that $d(M) < b(M)$, then (4.1) is better than (3.1). The matrix B of Example 4.1 provides an example of such a matrix, since $b(B) = n - 1$, while it is easily seen that $d(B) = 3$.

Recently, Hartwig and Neumann [5] have conjectured that if M is a real primitive matrix, then

$$\exp(M) \leqslant (m-1)^2 + 1, \tag{4.2}$$

where $m = m(M)$ is the degree of the minimal polynomial of M. Since $M_+^m \leqslant (I + M + \cdots + M^{m-1})_+$, it follows that $d \leqslant m - 1$ as long as M is irreducible. [There is no simple inequality relating $m(M)$ and $b(M)$, however.] This conjecture is weaker than (4.1) and has been verified in [5] for all but a few cases. As before, (3.1) is sometimes better than (4.2) and sometimes worse. For example, if the J blocks in the matrix M_1 of Table 1 are all 1×1 (or even if they are all square and of the same size), then [thinking of M_1 as a *real* (0, 1) matrix for the moment] $m(M_1) \leqslant b(M_1) + 1$. But $d(M_1) = b(M_1)$ here, so it follows that $m(M_1) = b(M_1) + 1$. Thus (3.1) is better than (4.2) in this case. On the other hand, let $M = I + H$, where H is an $n \times n$ skew-Hadamard design; that is, H is a real (0, 1) matrix such that

$$H + H^t = J - I \quad \text{and} \quad HH^t = \frac{n-3}{4}J + \frac{n+1}{4}I.$$

(Here real matrix arithmetic is used.) Then M is normal with three distinct eigenvalues, and so $m = 3$. However, the diagonal 1's of M_+ are isolated, so $b = n$. Consequently, the conjectured bound (4.2) is better than (3.1) in this case.

Our last comparison takes its cue from Proposition 3.1(b). Suppose that B is a primitive $(0,1)$ boolean matrix and that $B = XY$ is a boolean rank factorization of B. According to Proposition 3.1(b), $\exp(B) = \exp(YX) + \varepsilon$, where ε is 1, -1, or 0. The next result shows that all three possible values of ε can be realized when $\exp(YX)$ is maximum.

THEOREM 4.1. *Suppose that ε is either 1, -1, or 0 and that $2 \leqslant b \leqslant n-1$. There is a primitive $n \times n$ $(0,1)$ boolean matrix B with boolean rank b and a boolean rank factorization $B = XY$ such that $\exp(YX) = (b-1)^2 + 1$ and $\exp(B) = \exp(YX) + \varepsilon$.*

Proof. For $\varepsilon = 1$, consider any matrix B in Table 1 (or Table 2 if $b = 2$). Then $b(B) = b$ and $\exp(B) = (b-1)^2 + 2$ by Theorem 3.2. It follows from Propositions 3.1 and 1.1 that if $B = XY$ is any rank factorization of B, then $\exp(YX) = (b-1)^2 + 1$, so that $\exp(B) = \exp(YX) + \varepsilon$ with $\varepsilon = 1$.

For $\varepsilon = -1$, consider the $n \times n$ matrix

$$B = \left[\begin{array}{cccccc|ccc}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & \cdots & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & \cdots & 1 \\
\hline
0 & 1 & 0 & \cdots & 0 & 0 & & & \\
0 & 1 & 0 & \cdots & 0 & 0 & & & \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & 0_{n-b,n-b} & \\
0 & 1 & 0 & \cdots & 0 & 0 & & &
\end{array}\right].$$

Note that B has a set of isolated ones in positions $(b,1)$ and $(i, i+1)$, $1 \leqslant i \leqslant b-1$. Now let X be the $n \times b$ matrix

$$X = \left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 \\
1 & 0 & \cdots & 0 & 0 & 1 \\
\hline
1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0 & 0
\end{array}\right],$$

and let Y be the $b \times n$ matrix

$$Y = \left[\begin{array}{cccccc|ccc} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 1 \end{array}\right].$$

Then $XY = B$, so $b(B) \leqslant b$ and hence $b(B) = b$ by Proposition 2.1. Further, $YX = W_b$, so $\exp(YX) = b(b-1)^2 + 1$ by Proposition 1.1.

Let the principal submatrix of B on its first b rows and columns be C. Note that B can be written as

$$\left[\begin{array}{c|c} C & Ce_1 j_{n-b}^t \\ \hline j_{n-b} e_1^t C & C_{11} J_{n-b} \end{array}\right].$$

A straightforward proof by induction on k shows that for any $k \geqslant 1$,

$$B^k = \left[\begin{array}{c|c} C^k & C^k e_1 j_{n-b}^t \\ \hline j_{n-b} e_1^t C^k & C_{11}^{(k)} J_{n-b} \end{array}\right],$$

where $C_{11}^{(k)}$ is the $(1,1)$ entry of C^k. Thus we find that $\exp(B) = \exp(C)$. But C is known to have exponent $(b-1)^2$ (see Brualdi and Ryser [1, p. 83]). So we see that in this case, $\exp(B) = \exp(YX) + \varepsilon$ with $\varepsilon = -1$.

For $\varepsilon = 0$, consider the $n \times n$ matrix

$$\tilde{B} = \left[\begin{array}{cccccc|ccc} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ \hline 0 & 1 & 0 & \cdots & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & 0_{n-b,n-b} & \\ 0 & 1 & 0 & \cdots & 0 & 0 & & & \end{array}\right].$$

Again, $\tilde{B}$ has a set of b isolated ones, and $\tilde{B} = \tilde{X}Y$, where Y is as above and

$$\tilde{X} = \left[\begin{array}{cccc} & I_b & & \\ \hline 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{array}\right].$$

Thus by Proposition 2.1, $b(\tilde{B}) = b$. An argument similar to the one above shows that the exponent of $\tilde{B}$ is the same as that of its principal submatrix on the first b rows and columns. But that submatrix is W_b, so we have $\exp(\tilde{B}) = (b-1)^2 + 1$. However, $Y\tilde{X} = W_b$, so we see that in this case, $\exp(\tilde{B}) = \exp(Y\tilde{X}) + \varepsilon$ with $\varepsilon = 0$. ∎

5. PROBLEMS

In this section, we suggest a few problems arising from the ideas in the preceding sections.

(1) Let E_b be the set whose members are the exponents of the $b \times b$ primitive matrices, and let $E_{n,b}$ be the set whose members are the exponents of the $n \times n$ primitive matrices of boolean rank b. It is not difficult to show that $E_{n,b} \subseteq E_{n+1,b}$, and Theorem 3.1 implies that $\lim_{n\to\infty} E_{n,b}$ exists. Can this limit set be described more explicitly? Perhaps in terms of E_b?

(2) Fix an $n \geqslant 3$, a b with $2 \leqslant b \leqslant n-1$, and an ε equal to either 1, -1, or 0. Given $k \in E_b$, is there an $n \times n$ primitive matrix B with $b(B) = b$ which has a boolean rank factorization $B = XY$ such that $\exp(YX) = k$ and $\exp(B) = k + \varepsilon$? Theorem 4.1 answers the question in the affirmative when $k = (b-1)^2 + 1$, but what about other values of k?

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